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Invariance relations for laminar forced convection in ducts with slowly varying cross-section

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Abstract

Laminar flow forced convective heat transfer in ducts with slowly varying cross-section is analysed. The analysis is based on a simplifying assumption concerning the velocity field, called the similarity assumption. It is assumed that the axial velocity profile is locally fully developed, irrespective of the variation of the cross-sectional area with axial position. It is shown that in the axisymmetric case (circular tube with varying diameter) for boundary conditions of the first kind the diameter variation has no influence at all on the heat transferred. As in the case of a tube with constant diameter, the degree of temperature equilibration depends only on the Fourier number

$$Fo = \frac{\pi a L}{Q}$$

that is on the ratio of length of the tube, L, to flow rate Q. In the case of a plane channel, however, variations of the channel height $2H$ have a pronounced influence on the transfer coefficient. It is shown that the degree of equilibration depends only on the effective Fourier number

$$Fo_{\text{eff}} = \frac{aB}{Q} \cdot \int_0^L \frac{\mathrm{d}z}{H(z)}.$$

The heat transfer is enhanced by a narrow design of the duct. These results hold for any shape of the axial velocity distribution over the duct cross-section, provided the above-mentioned similarity assumption is justified.

By analogy, all foregoing results are also applicable to the convective–diffusive mass transfer in flow through ducts with slowly varying cross-section.

The range of validity of the similarity assumption is discussed in a hydrodynamic section. It can be shown that for creeping flow of a Newtonian liquid in an infinite cone or wedge this assumption is a very good approximation to the exact solution of Stokes' equations for cone or wedge angles up to 30°. Apart from Newtonian creeping flow, the assumption might well be applicable in the duct flow of a power-law fluid and in pure elongational flow as encountered in stretching filaments or sheets of liquid, where the axial velocity is constant over the cross-section.

1. Introduction

Laminar flow forced convection in ducts has been a subject of engineering research since the fundamental papers of both Graetz [1] and Nusselt [2]. The thermal entrance problem for hydrodynamically developed flow in a circular tube, first tackled independently by those authors and later by several others, is nowadays known as the *Graetz problem* or *Graetz–Nusselt problem* [3]. Numerous extensions or modifications of the problem have been treated in the last few decades. Examples are a variety of boundary conditions, the effect of axial conduction, non-Newtonian fluid flow and duct cross-sections other than circular. In particular, fully-developed flow between parallel plates (slit flow) is of great

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Nomenclature

a (m^2/s)	thermal diffusivity of liquid
B (m)	channel width
F (dimensionless)	dimensionless stream function (axisymmetric flow)
f (dimensionless)	dimensionless velocity profile function (axisymmetric flow)
G (dimensionless)	dimensionless stream function (plane flow)
g (dimensionless)	dimensionless velocity profile function (plane flow)
H (m)	half of channel height
L (m)	duct length
p (Pa)	pressure
Q (m^3/s)	volumetric flow rate
q (W/m^2)	heat flux density
R (m)	tube radius
r (m)	radial co-ordinate
T (K)	temperature
$\bar{T}$ (K)	(flow average) bulk temperature
u, v, w (m/s)	velocity components
$\bar{w}$ (m/s)	average axial velocity
x, y, z (m)	Cartesian co-ordinates
Fo (dimensionless)	Fourier number
Nu (dimensionless)	Nusselt number
Re (dimensionless)	Reynolds number

Greek symbols

α (dimensionless)	wall inclination angle
Δ ($1/m^2$)	Laplacian operator
ρ (m)	radius (spherical polar co-ordinates)
ϑ (dimensionless)	angle (spherical polar co-ordinates)
λ (W/m^2 K)	thermal conductivity
μ (Pa s)	dynamic viscosity
Θ (dimensionless)	dimensionless temperature
$\bar{\Theta}$ (dimensionless)	dimensionless bulk temperature
ω (dimensionless)	angle (cylindrical polar co-ordinates)
Ψ (m^3/s)	stream function (axisymmetric flow)
Φ (m^2/s)	stream function (plane flow)
η, ζ (dimensionless)	dimensionless co-ordinates

Subscripts

0	entrance value
w	value at duct wall
eff	effective value
loc	local value
SA	according to the similarity assumption

practical importance. An overview of work on these topics for Newtonian flow up to 1978 is given by the monograph of Shah and London [4]. Solutions for non-Newtonian flow can be found in [5,6].

The objective of this paper is to analyse the influence of a *varying duct cross-section* on the laminar convective–conductive heat transfer. This subject, though of little relevance for conventional heat exchangers, is of particular interest in the polymer producing and processing industries, since many polymer melt flow systems comprise nozzles, dies and manifolds in which the above-mentioned conditions are applicable. Examples for such flow geometries are sheet extrusion dies, underwater pelletiser die-heads, spinnerets, etc. In most of these applications high viscosity polymer melt flow in comparatively small scale ducts is encountered. Hence, the analysis may be restricted to nearly zero Reynolds number (creeping) flow, thus omitting the inertia terms in the momentum balance equations.

Two fundamental flow systems are considered in this paper: Flow in a circular tube with slowly varying diameter and plane channel flow with slowly varying channel height. In order to facilitate an analytical treatment of the convection–conduction problem, a simplifying assumption concerning the flow kinematics is used. This assumption is introduced in the following Section 2 without further fluid-mechanical justification. Sections 3 and 4 deal with the consequences for the convective–conductive heat transfer in ducts with varying cross-sectional area. Finally, in Section 5, as a justification for the kinematic assumption, its range of validity is checked by means of comparison with exact analytical solutions for Newtonian creeping flow in a semi-infinite cone or wedge.

2. Kinematics of flow in ducts with non-uniform cross-section

2.1. Circular tube with slowly varying diameter

Consider the laminar flow of an incompressible viscous liquid in a circular tube with slowly varying diameter (Fig. 1). From the fluid-mechanical point of view, it seems reasonable to assume that the axial velocity component approximates the fully-developed velocity profile to a certain degree of accuracy, depending on the rheological properties of the fluid, the Reynolds number and the duct contour function $R(z)$. This issue has been discussed, for instance, by Batchelor for the Newtonian case [7]. A detailed enquiry on the accuracy of this assumption for creeping flow of a Newtonian liquid is given in Section 5.

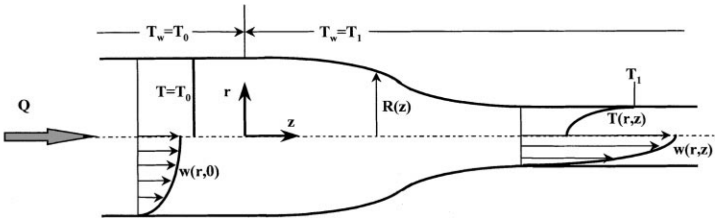

Fig. 1. An illustration of the convection–conduction problem in ducts with varying cross-section.

Let $f(\eta)$ be the shape function of the fully-developed axial velocity profile in a tube with constant radius R_0, that is

$$w(r) = \bar{w} \cdot f\left(\frac{r}{R_0}\right), \tag{1}$$

where $\bar{w} = Q/\pi R_0^2$ denotes the mean axial velocity and $\eta = r/R_0$ is a reduced radial co-ordinate. The similarity assumption for flow in a tube with varying cross-section means that the axial velocity profile has the same shape as in the constant diameter case [7]

$$w(r,z) = \bar{w}(z) \cdot f\left(\frac{r}{R(z)}\right), \tag{2}$$

where the mean axial velocity depends on z: $\bar{w}(z) = Q/\pi[R(z)]^2$. The radial velocity component may be calculated by integrating the continuity equation. In order to simplify the derivation, the stream function for axisymmetric flow, $\Psi(r,z)$, is introduced, thus satisfying the continuity equation identically:

$$w(r,z) = \frac{1}{r}\frac{\partial \Psi}{\partial r}, \tag{3a}$$

$$v(r,z) = -\frac{1}{r}\frac{\partial \Psi}{\partial z}. \tag{3b}$$

Formulated in terms of the stream function, the similarity assumption reads

$$\Psi(r,z) = \frac{Q}{\pi} \cdot F\left(\frac{r}{R(z)}\right), \tag{4}$$

where $F(\eta)$ is the dimensionless stream function for fully-developed flow in a tube with constant diameter. The specific shape of $F(\eta)$ depends on the rheological properties of the fluid and the boundary conditions at the tube wall. Using the defining Eqs. (3a) and (3b) it is found from Eq. (4) that

$$w(r,z) = \frac{Q}{\pi} \cdot \frac{1}{rR(z)} \cdot F'(\eta) = \bar{w}(z)\frac{F'(\eta)}{\eta} \tag{5a}$$

and

$$v(r,z) = \frac{Q}{\pi} \cdot \frac{R'(z)}{[R(z)]^2} \cdot F'(\eta) = \bar{w}(z)R'(z)F'(\eta). \tag{5b}$$

The prime applied to a function of a single variable denotes a derivative with respect to that variable. Comparing the right-hand sides of Eqs. (5a) and (5b) a general relation between the radial and the axial velocity component is found, which holds irrespective of the specific choice of the stream function $F(\eta)$

$$v(r,z) = \eta R'(z) w(r,z). \tag{6}$$

As a consequence of the similarity assumption, the radial velocity component is proportional to the axial velocity component, the reduced radial co-ordinate η and the tangent of the wall inclination angle $\tan\alpha = \mathrm{d}R/\mathrm{d}z$. The relation between the axial velocity shape function $f(\eta)$ and the dimensionless stream function $F(\eta)$ is

$$f(\eta) = \frac{F'(\eta)}{\eta}. \tag{7}$$

As an example, Fig. 2 shows the stream lines for Newtonian flow in a tube with axially varying cross-section, calculated by means of the similarity assumption. Each stream line can be constructed from the wall contour function $R(z)$ by multiplication with the appropriate value of the reduced radial co-ordinate η.

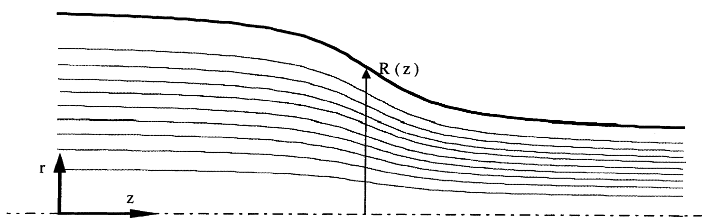

Fig. 2. Stream-lines in a tube with non-uniform diameter, calculated by means of the similarity assumption for Newtonian flow, i.e. parabolic distribution of the axial velocity in each cross-section.

2.2. Plane channel with slowly varying height

In the following, it is assumed that the axial velocity component is symmetric with respect to $y = 0$, that is $w(-y, z) = w(y, z)$. Introducing a stream function $\Phi(y, z)$ for two-dimensional plane flow which is defined by the relations

$$w(y,z) = \frac{\partial \Phi}{\partial y}, \tag{8a}$$

$$v(y,z) = -\frac{\partial \Phi}{\partial z} \tag{8b}$$

the similarity assumption can be formulated as

$$\Phi(y,z) = \frac{Q}{2B} \cdot G\left(\frac{y}{H(z)}\right), \tag{9}$$

where $G(\eta)$ is a dimensionless stream function which depends solely on the reduced lateral co-ordinate $\eta = y/H(z)$. The channel height is $2H(z)$. Q/B denotes the volumetric flow rate per unit width of the channel. Combined with the symmetry assumption, this definition of $G(\eta)$ ensures that $G(1) - G(0) = G(0) - G(-1) = 1$. Calculating the velocity components from Eqs. (8a)–(9), one finds:

$$w(y,z) = \frac{Q}{2B} \cdot G'(\eta) \cdot \frac{1}{H(z)} = \bar{w}(z) G'(\eta) = \bar{w}(z) g(\eta), \tag{10a}$$

$$v(y,z) = \frac{Q}{2B} \cdot G'(\eta) \cdot \frac{yH'(z)}{[H(z)]^2} = \bar{w}(z) H'(z) \eta g(\eta). \tag{10b}$$

In Eqs. (10a) and (10b) the mean axial velocity $\bar{w}(z) = Q/2BH(z)$ has been introduced, as well as a shape function $g(\eta) = G'(\eta)$ which describes the distribution of the axial velocity component in dimensionless form. Because of the assumed symmetry $g(-\eta) = g(\eta)$. A comparison of Eqs. (10a) and (10b) yields the relation

$$v(y,z) = \eta H'(z) w(y,z) \tag{11}$$

which is identical to the tube case.

3. Laminar forced convection in a circular tube with slowly varying diameter

Neglecting axial conduction and heat sources (viscous dissipation, chemical reactions) and assuming constant thermal properties and rotational symmetry, the energy equation can be written

$$w(r,z)\frac{\partial T}{\partial z} + v(r,z)\frac{\partial T}{\partial r} = \frac{a}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right), \tag{12}$$

where a denotes the thermal diffusivity of the liquid. The velocity components are given by Eqs. (2) and (6). Assuming a uniform entrance temperature T_0 and a sudden jump of the wall temperature from T_0 to T_w at the axial position $z = 0$, the following boundary conditions hold:

$$z = 0, \quad 0 \leqslant r \leqslant R_0 \quad \Rightarrow \quad T = T_0, \tag{13a}$$

$$z > 0, \quad r = R(z) \quad \Rightarrow \quad T = T_w, \tag{13b}$$

$$z > 0, \quad r = 0 \quad \Rightarrow \quad \frac{\partial T}{\partial r} = 0. \tag{13c}$$

Note that the following derivation is also valid for generalized boundary conditions of the first kind, that is for a prescribed wall temperature profile $T_w = T_w(z)$. For the sake of simplicity, the derivation shall be given only for a wall temperature jump at $z = 0$.

At first glance the system of Eqs. (2), (6), (12)–(13c) seems rather complicated. However, as has been pointed out before by the author [8] for the special case $f(\eta) \equiv 1$, a significant simplification may be achieved by a suitable transformation. Introducing the new variables

$$\eta(r,z) := \frac{r}{R(z)}, \tag{14a}$$

$$\Theta(\eta,z) := \frac{T(r,z) - T_w}{T_0 - T_w} \tag{14b}$$

into Eq. (12) it is found that

$$w(r,z)\left(\frac{\partial \Theta}{\partial z} + \frac{\partial \Theta}{\partial \eta}\frac{\partial \eta}{\partial z}\right) + v(r,z)\frac{\partial \Theta}{\partial \eta}\frac{\partial \eta}{\partial r} = \frac{a}{\eta}\frac{\partial}{\partial \eta}\left(\eta\frac{\partial \Theta}{\partial \eta}\right)\left(\frac{\partial \eta}{\partial r}\right)^2. \tag{15}$$

Upon substitution of the partial derivatives of η with respect to r and z and use of relation Eq. (6) between the two velocity components, it is found that the radial convective term in Eq. (15) is cancelled out, yielding the following simplified equation:

$$w(r,z)[R(z)]^2\frac{\partial \Theta}{\partial z} = \frac{a}{\eta}\frac{\partial}{\partial \eta}\left(\eta\frac{\partial \Theta}{\partial \eta}\right). \tag{16}$$

The elimination of the radial convective term could be anticipated: With the assumed kinematics, the stream lines are characterised by constant values of η. This implies that η is a material or Lagrangian co-ordinate. No convective transport terms may occur within a Lagrangian temperature field description. However, if axial conduction had to be accounted for, the simplification of the convective terms would have been outweighed by a complication of the conductive terms. Apart from fluid-mechanical considerations this is another reason to restrict the analysis to small wall inclination angles α.

Using Eq. (2), the product wR^2 in Eq. (16) may be expressed by the volumetric flow rate Q and the velocity shape function $f(\eta)$:

$$w(r,z)[R(z)]^2 = \frac{Q}{\pi} f(\eta). \tag{17}$$

This completes the transformation to the new independent variables. Finally, introducing a new dimensionless axial co-ordinate

$$\zeta := \frac{\pi a z}{Q} \tag{18}$$

the heat equation can be written

$$f(\eta)\frac{\partial \Theta}{\partial \zeta} = \frac{1}{\eta}\frac{\partial}{\partial \eta}\left(\eta \frac{\partial \Theta}{\partial \eta}\right). \tag{19}$$

The dimensionless axial co-ordinate ζ is a Fourier or reciprocal Graetz number. In most of the current literature the Graetz number $Gz = Q/(\pi a z)$ is applied for forced convection in ducts whereas the Fourier number is used in unsteady heat transfer problems to define a dimensionless time. However, the author prefers to apply the Fourier number also in duct heat transfer calculations. This habit emphasises the close familiarity between unsteady heat or mass transfer problems and steady laminar forced convection. This familiarity has been stressed in an earlier paper [8]. Readers who reject the use of a Fourier number in forced convection problems are asked to understand this term as a technical abbreviation for the reciprocal value of the Graetz number.

The boundary conditions, written in terms of the dimensionless variables Θ, η and ζ, are:

$$\zeta = 0, \qquad 0 \leqslant \eta \leqslant 1 \quad \Rightarrow \quad \Theta = 1, \tag{20a}$$

$$\zeta > 0, \qquad \eta = 1 \quad \Rightarrow \quad \Theta = 0, \tag{20b}$$

$$\zeta > 0, \qquad \eta = 0 \quad \Rightarrow \quad \frac{\partial \Theta}{\partial \eta} = 0. \tag{20c}$$

Surprisingly, in differential Eq. (19) and boundary Eqs. (20a)–(20c) there is no more explicit dependence on the tube contour function $R(z)$. Those equations merely constitute the dimensionless form of a generalised Graetz problem for flow in a tube with uniform diameter. (The attribute 'generalised' refers to the arbitrary axial velocity profile $f(\eta)$ in contrast to the parabolic velocity profile of the 'classical' Graetz problem.) From this result, the following invariance property for laminar forced convective transport in axisymmetric flow with varying cross-section can be deduced:

Let $\Theta_f\left(\frac{r}{R_0}, \frac{\pi a z}{Q}\right)$ be the non-dimensionalised solution[1] of the generalised Graetz problem for a tube with uniform radius R_0. Then the solution to the analogous problem with varying tube radius $R(z)$ is given by $\Theta_f\left(\frac{r}{R(z)}, \frac{\pi a z}{Q}\right)$, provided the velocity field obeys the similarity assumption.

In particular, the dimensionless bulk temperature is invariant with respect to the tube contour function $R(z)$ and solely dependent on the tube Fourier number. To realise this, consider the definition of the bulk (flow average) temperature

$$\bar{T}_f(z) := \frac{1}{Q} \cdot \int_0^{R(z)} T_f(r,z) w(r,z) 2\pi r \, \mathrm{d}r. \tag{21}$$

Transforming Eq. (1) to the dimensionless quantities yields

$$\bar{\Theta}_f(\zeta) = \int_0^1 \Theta_f(\eta,\zeta) f(\eta) 2\eta \, \mathrm{d}\eta \tag{22}$$

which is independent of the tube contour function $R(z)$. Hence, any published results for the dependence of the bulk temperature on the Fourier (or Graetz) number can be generalised as well for those cases where the tube radius is weakly non-uniform and the flow field may be described by the similarity assumption.

Finally, the invariance relation shall be formulated in terms of the local and average tube Nusselt numbers. Consider the wall heat flux density for any velocity shape function f and arbitrary tube contour function $R(z)$

$$\begin{aligned} q_{\mathrm{w},f} &= -\lambda \left.\frac{\partial T_f(r,z)}{\partial r}\right|_{r=R(z)} \\ &= -\lambda \frac{T_0 - T_w}{R(z)} \left.\frac{\partial \Theta_f(\eta,\zeta)}{\partial \eta}\right|_{\eta=1}. \end{aligned} \tag{23}$$

Obviously, the wall heat flux density is inversely proportional to the local tube radius. However, if the *local* Nusselt number is defined with the *local* tube radius, it turns out to be invariant with respect to the tube contour function

$$Nu_{\mathrm{loc},f} = \frac{q_{\mathrm{w},f} R(z)}{\lambda\left(\bar{T}_f(z) - T_\mathrm{w}\right)} = -\frac{1}{\bar{\Theta}_f(\zeta)} \left.\frac{\partial \Theta_f(\eta,\zeta)}{\partial \eta}\right|_{\eta=1}. \tag{24}$$

From an over-all heat balance for a tube section of length dz the following relation can be deduced [9]

$$\left.\frac{\partial \Theta_f(\eta,\zeta)}{\partial \eta}\right|_{\eta=1} = \frac{1}{2} \cdot \frac{\mathrm{d}\bar{\Theta}_f(\zeta)}{\mathrm{d}\zeta}. \tag{25}$$

Inserting Eq. (25) into Eq. (24), the average Nusselt number can be calculated as

$$Nu_f = \frac{1}{\zeta} \cdot \int_0^\zeta Nu_{\mathrm{loc},f} \, \mathrm{d}\zeta = -\frac{1}{2\zeta} \cdot \ln\left(\bar{\Theta}_f(\zeta)\right). \tag{26}$$

This means that any result given in the literature for the average Nusselt number in a tube of constant radius for

[1] Throughout this section, the subscript 'f' is used as a reminder that the quantity attributed with it is a *functional* of the particular velocity shape function $f(\eta)$ considered.

boundary conditions of the first kind holds as well for a slowly varying tube radius, provided the similarity assumption is applicable.

Duct heat transfer results may be presented in terms of either the average Nusselt number or the bulk temperature as a function of the axial co-ordinate. Depending on the practical purpose, either of the representations may be more advantageous. In this paper, the $\bar{\Theta}$ versus ζ plot is preferred because it is considered to describe the temperature equilibration in a more obvious way.

Two special cases for the axial velocity shape function are of outstanding practical importance. The first one is the plug flow case

$$f(\eta) \equiv 1 \tag{27}$$

and the second one is the Hagen–Poiseuille case (parabolic velocity profile)

$$f(\eta) = 2(1 - \eta^2). \tag{28}$$

Though plug flow is less probable in a tube with non-uniform radius, it plays an important role in flow situations without solid boundaries, as encountered, for instance, in liquid jets and the fibre spinning process. In such applications, researchers frequently agree upon the assumption that the axial velocity becomes independent of the radial co-ordinate some diameters downstream of the jet nozzle or spinneret. The type of deformation encountered under these circumstances is almost pure uniaxial elongation. This case has been treated before by the author [8].

The parabolic velocity profile Eq. (28) occurs in the fully-developed tube flow of Newtonian liquids. In a tube with axially varying diameter the velocity profile Eq. (28) can no longer be valid exactly, the degree of deviation being dependent on the shape of the tube contour function and the Reynolds number. This issue will be discussed in more detail in Section 5.

Because of their practical importance the dimensionless bulk temperature is plotted for both special cases in Fig. 3 as a function of the tube Fourier number $Fo = \pi aL/Q$. The solution for plug flow can be taken from standard textbooks on heat transfer in solid bodies. Eqs. (19)–(20c) with a flat velocity profile Eq. (27) merely constitute the mathematical formulation for the unsteady heating or cooling of an infinite circular cylinder. The solution for parabolic flow was calculated as an exponential series using the eigenvalues and constants given by Brown [10].

Note on boundary conditions: Throughout this section, uniform wall and entrance temperatures have been assumed, their difference being the driving force for the heat transfer process. It is, of course, seductive to generalise the above invariance relation to different thermal boundary conditions. However, one readily finds that any boundary condition referring to the wall heat flux will fail to be transformed invariantly because the wall temperature gradient is inversely proportional to the local tube radius. Only in the case of an adiabatic tube wall (homogeneous boundary condition of the second kind) the invariance with respect to the tube radius holds. On the other hand, it is clear that the entrance temperature distribution is irrelevant for the success of our transformation. Thus it can be deduced that the equilibration of a non-uniform entrance temperature distribution within an ideally isolated tube takes place on the same tube length scale whatever the tube contour function $R(z)$.

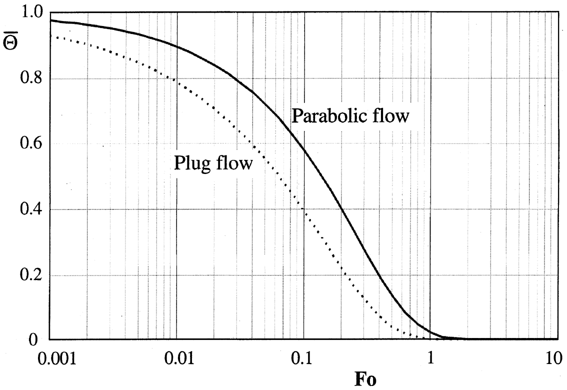

Fig. 3. Dimensionless bulk (flow-average) temperature as a function of the tube Fourier number, calculated for the plug flow and the parabolic flow cases.

4. Laminar forced convection in a plane channel with slowly varying height

Neglecting axial conduction and heat sources, the heat equation is written in Cartesian co-ordinates as

$$w(y,z)\frac{\partial T}{\partial z} + v(y,z)\frac{\partial T}{\partial y} = a\frac{\partial^2 T}{\partial y^2} \tag{29}$$

with the following boundary conditions of the first kind:

$$z = 0, \qquad -H_0 \leqslant y \leqslant H_0 \quad \Rightarrow \quad T = T_0, \tag{30a}$$

$$z > 0, \qquad y = \pm H(z) \quad \Rightarrow \quad T = T_w. \tag{30b}$$

As in the tube case, the system of Eqs. (29)–(30b) is transformed to a material lateral co-ordinate η and a non-dimensional temperature:

$$\eta(y,z) = \frac{y}{H(z)}, \tag{31a}$$

$$\Theta(\eta,z) = \frac{T(y,z) - T_w}{T_0 - T_w}. \tag{31b}$$

The lateral convective term is eliminated by this transformation. The resulting differential equation is

$$w(y,z)[H(z)]^2\frac{\partial\Theta}{\partial z}=a\frac{\partial^2\Theta}{\partial\eta^2}. \tag{32}$$

Using the continuity relation

$$w(y,z)H(z)=\frac{Q}{2B}g(\eta). \tag{33}$$

Eq. (32) can be written

$$\frac{QH(z)}{2B}g(\eta)\frac{\partial\Theta}{\partial z}=a\frac{\partial^2\Theta}{\partial\eta^2}. \tag{34}$$

Comparing Eq. (34) with Eq. (19), a difference becomes evident between the tube and the channel case. There remains an explicit channel height dependence in the heat equation after the transformation. This is due to the differences in the equations describing over-all continuity for the two cases. A stretched axial co-ordinate is defined in order to simplify the differential Eq. (34)

$$\zeta(z)=\frac{2aB}{Q}\int_0^z\frac{\mathrm{d}z}{H(z)}. \tag{35}$$

This leads to

$$g(\eta)\frac{\partial\Theta}{\partial\zeta}=a\frac{\partial^2\Theta}{\partial\eta^2}. \tag{36}$$

The boundary conditions, written in terms of the dimensionless variables Θ,η and ζ need not be repeated here, since they are analogous to the dimensionless boundary conditions derived in the tube case, Eqs. (20a)–(20c).

Eq. (36) and transformed boundary conditions Eqs. (30a) and (30b) constitute the dimensionless representation of a generalised Graetz problem [2] for a plane channel with uniform height. Hence, every solution derived for a specific axial velocity profile $g(\eta)$ and constant channel height may be used as well for the variable height case. This invariance property may be formulated as follows:

Let $\Theta_g\left(\frac{y}{H_0},\frac{2aBz}{QH_0}\right)$ be the non-dimensionalised solution [3] of the generalised Graetz problem for a plane channel with uniform height $2H_0$. Then the solution to the analogous problem with non-uniform channel height $2H(z)$ is given by $\Theta_g\left(\frac{y}{H(z)},\frac{2aB}{Q}\cdot\int_0^z\frac{\mathrm{d}z}{H(z)}\right)$, provided the similarity assumption concerning the flow kinematics is applicable.

The stretched axial co-ordinate Eq. (35) may be interpreted as an *effective Fourier number* (or reciprocal effective Graetz number) which accounts for the effect of a non-uniform channel height on the heat transfer process.

It follows that the bulk or flow-average temperature

$$\bar{T}_g(z):=\frac{2B}{Q}\cdot\int_0^{H(z)}T_g(y,z)w(y,z)\,\mathrm{d}y \tag{37}$$

may be calculated as in the constant height case by using the effective Fourier number Eq. (35):

Let $\bar{\Theta}_g\left(\frac{2aB}{Q}\frac{z}{H_0}\right)$ be the dimensionless bulk temperature in plane channel flow for a particular choice of the axial velocity profile $g(\eta)$ and uniform channel height $2H_0$. The dimensionless bulk temperature for a varying channel height is then given by $\bar{\Theta}_g\left(\frac{2aB}{Q}\cdot\int_0^z\frac{\mathrm{d}z}{H(z)}\right)$ provided the similarity assumption holds true.

It is well known that laminar convective heat transfer in a uniform plane channel can be enhanced by decreasing the channel height $2H_0$. For given constant values of length L, width B and throughput Q this measure increases the channel Fourier number $Fo=2aBL/(QH_0)$. The invariance relation stated above may be considered as a generalisation of that result for the case of a non-uniform channel height. It gives a method for incorporating a given height contour function $H(z)$ into heat transfer calculations. For given values of length, width and throughput and a constant channel wall temperature, channels with different height contour functions $H(z)$ are thermally equivalent if the areas under the $1/H(z)$-curves are equal. This implies, for example, that opposing the direction of flow in an isothermal channel with non-uniform height does not alter the total heat flux between the fluid and the channel walls.

As in the tube case, two specific choices of the velocity shape function are of outstanding practical importance. These are plug flow

$$g(\eta)\equiv 1 \tag{38}$$

and parabolic flow

$$g(\eta)=\frac{3}{2}(1-\eta^2). \tag{39}$$

Possible applications for the plug flow solution might be polymer film processing, whereas the parabolic case occurs in plate or slit heat exchangers and in film extrusion dies. The functional dependence of the dimensionless bulk temperature on the effective Fourier number is reproduced graphically for those two cases in Fig. 4. The solution for plug flow has been taken from a standard textbook. This case is mathematically equivalent with the unsteady heating or cooling of a solid slab. The solution for parabolic flow has been calculated using the eigenvalues and constants given by Brown [10].

[2] Problem generalised from parabolic (Newtonian) axial velocity profile to an arbitrary velocity profile, described by the shape function $g(\eta)$.

[3] The subscript 'g' is used as a reminder that the quantity attributed with it is a *functional* of $g(\eta)$.

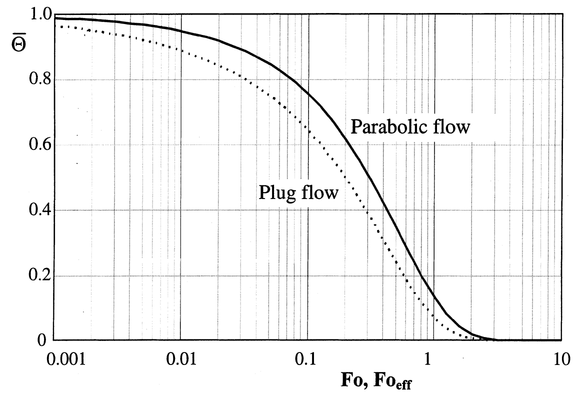

Fig. 4. Dimensionless bulk (flow-average) temperature as a function of the effective channel Fourier number, calculated for the plug flow and the parabolic flow cases.

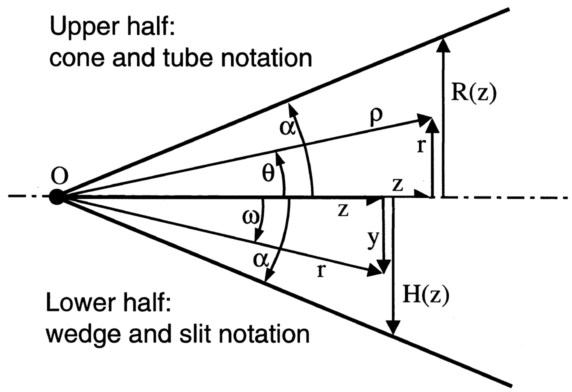

Fig. 5. Co-ordinate notation used in the comparison with exact solutions for flow in a tube with non-uniform radius (upper half) and flow in a plane channel with non-uniform height (lower half).

Note on boundary conditions: Apart from boundary conditions of the first kind, as applied above, the invariance relation holds also for homogeneous boundary conditions of the second kind, that is for one or both of the channel walls being adiabatic. The combination of an isothermal and an adiabatic duct wall may be of practical importance in some polymer flow systems where only one side of a flow channel can be reached by a thermostatting device.

5. Fluid-mechanical justification for the similarity assumption

So far, only the consequences of the similarity assumption on convective–conductive transport in ducts with non-uniform cross-section have been considered. This section is an attempt to demonstrate the degree of accuracy of that assumption for a given flow situation. For the sake of simplicity, the analysis shall be restricted to nearly zero Reynolds number (creeping) flow conditions, that is, inertia terms in the momentum equations are considered negligible compared with the viscous terms.

In order to facilitate an analytical treatment, it is furthermore assumed that the melt is Newtonian with constant viscosity. The flow is then described by Stokes' equations

$$0 = -\mathrm{grad}\, p + \mu \Delta \vec{v}, \tag{40}$$

where p denotes the pressure, μ the dynamic viscosity, $\vec{v}$ the velocity vector and Δ is the Laplacian operator.

No general solution of Stokes' equations is known for the flow in a duct with arbitrarily varying cross-section. However, simple solutions may be derived for the cases where the wall inclination angle is constant, those cases describing flow in a semi-infinite cone or wedge, respectively. The idea of this section is to compare those exact solutions with the similarity assumption. The degree of deviation between the flow fields calculated by both methods gives an idea of the accuracy of the similarity assumption, although higher order effects such as the influence of the curvature of the contour function cannot be incorporated herein.

Due to the nature of the four flow situations considered in this section, there is some risk of getting confused with co-ordinate systems. Fig. 5 is intended to visualise the notation applied in the different cases. The upper half shows how co-ordinates are interrelated in the comparison between the exact solution for flow in an infinite cone and the approximate solution for flow in a tube with non-uniform radius. The lower half demonstrates the analogous issue for the flow in a plane channel with non-uniform height.

Table 1 gives an overview of co-ordinate systems and the notation for the velocity components used in this paper.

The comparison between exact and approximate solutions is actually performed more easily on the stream functions rather than the velocity fields. Thus, confusion with co-ordinate systems and velocity components can be avoided.

5.1. Stokes flow in an infinite cone

Consider the creeping flow of a Newtonian liquid in a semi-infinite cone (Fig. 5, upper half of drawing). At the apex of the cone, a point source of strength Q is assumed. In spherical polar co-ordinates $(\rho, \vartheta, \omega)$, an axisymmetric solution to Stokes' equations satisfying the no-slip boundary condition at the cone walls $\vartheta = \pm\alpha$ is given by the velocity field [11]

$$v_\rho = \frac{3Q}{2\pi\rho^2} \cdot \frac{\cos^2\vartheta - \cos^2\alpha}{1 - 3\cos^2\alpha + 2\cos^3\alpha}, \tag{41a}$$

Table 1
Overview of the notation for co-ordinate systems and velocity components used in this paper

Original problem	Flow in a circular tube with non-uniform radius	Flow in a channel with non-uniform height
Co-ordinate system used	Cylindrical polar	Cartesian
	$r,\ \omega,\ z$	$x,\ y,\ z$
Velocity components	$v,\ -,\ w$	$-,\ v,\ w$
Analogy for comparison	Flow in a semi-infinite cone	Flow in a semi-infinite wedge
Co-ordinate system used in the analogy	Spherical polar	Cylindrical polar
	$\rho,\ \theta,\ \omega$	$r,\ \omega,\ z$
Velocity components in the analogy	$v_\rho,\ v_\theta,\ -$	$v_r,\ v_\omega,\ -$

$$v_\vartheta = 0. \tag{41b}$$

The stream function, defined by

$$v_\rho = -\frac{1}{\rho^2 \sin\vartheta} \cdot \frac{\partial \Psi}{\partial \vartheta}, \tag{42a}$$

$$v_\vartheta = \frac{1}{\rho \sin\vartheta} \frac{\partial \Psi}{\partial \rho} \tag{42b}$$

may be calculated by integrating Eqs. (41a) and (41b) as

$$\Psi(\vartheta) = \frac{Q}{2\pi} \cdot \frac{3 \cdot (1-\cos\vartheta)\cos^2\alpha + \cos^3\vartheta - 1}{1 - 3\cos^2\alpha + 2\cos^3\alpha}. \tag{43}$$

This exact solution has to be compared to the flow field given by the similarity assumption.

5.2. Stokes flow in an infinite wedge

Consider the creeping flow of a Newtonian liquid in a semi-infinite wedge (Fig. 5, lower half of drawing). At the apex of the wedge, a line source or sink of strength Q/B is assumed (volumetric flow rate per unit width of the wedge). A solution to Stokes' equations in cylindrical polar co-ordinates (r, ω) satisfying the no-slip boundary condition at the wedge walls $\omega = \pm\alpha$ is given by the velocity field [11]:

$$v_r = \frac{1}{r} \cdot \frac{\partial \Phi}{\partial \omega} = \frac{Q}{Br} \cdot \frac{\cos(2\omega) - \cos(2\alpha)}{\sin(2\alpha) - 2\alpha\cos(2\alpha)}, \tag{44a}$$

$$v_\omega = -\frac{\partial \Phi}{\partial r} = 0. \tag{44b}$$

The stream function may be calculated by integrating Eqs. (44a) and (44b) as:

$$\Phi(\omega) = \frac{Q}{2B} \cdot \frac{\sin(2\omega) - 2\omega\cos(2\alpha)}{\sin(2\alpha) - 2\alpha\cos(2\alpha)}. \tag{45}$$

Both exact solutions (41a), (41b) and (44a), (44b) describe a motion upon straight stream-lines with a common intersection at the origin.

Streamline patterns calculated under the similarity assumption have, of course, the same features. However, the distances between streamlines will be different, the latter difference being a measure for the accuracy of the assumption. Using cartesian co-ordinates, the wedge wall contour function is given by

$$H(z) = z \cdot \tan\alpha. \tag{46}$$

Hence the reduced lateral co-ordinate can be written

$$\eta = \frac{y}{H(z)} = \frac{\tan\omega}{\tan\alpha}. \tag{47}$$

Using the similarity assumption, the stream function for flow in the wedge is given by

$$\Phi_{SA}(\eta) = \frac{Q}{2B} \cdot \frac{3}{2} \cdot \eta \cdot \left(1 - \frac{1}{3}\eta^2\right). \tag{48}$$

The dimensionless stream function $G = 2B\Phi/Q$ calculated from Eqs. (45) and (48) is plotted in Fig. 6 versus the reduced lateral co-ordinate η for several values of the wedge half-angle α. Up to a half-angle of 15° only small

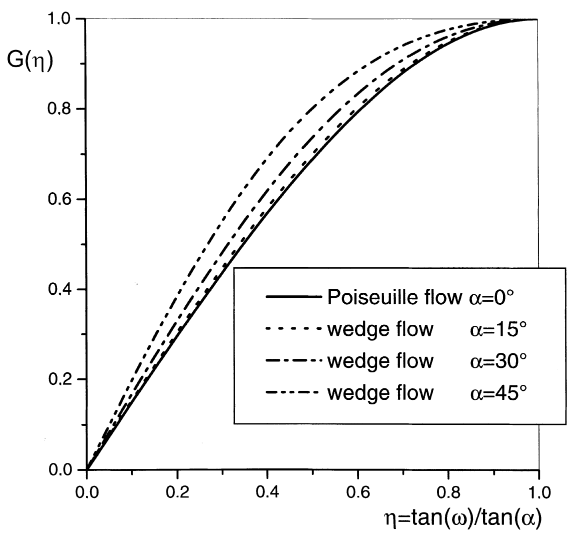

Fig. 6. Plot of the dimensionless stream function for flow in a semi-infinite wedge, calculated from the exact solution of Stokes' equations and according to the similarity assumption ('Poiseuille flow') for different values of the wedge half-angle α.

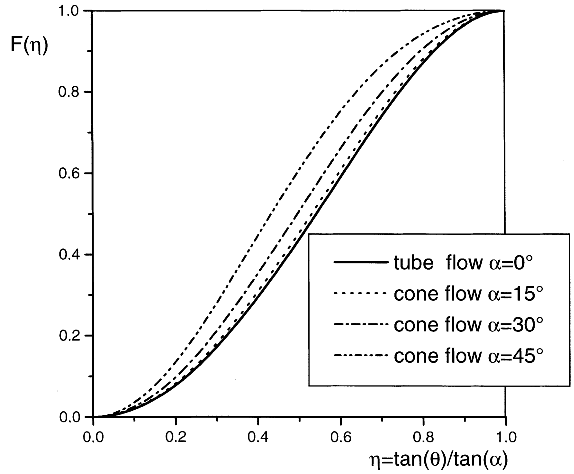

Fig. 7. Plot of the dimensionless stream function for flow in a semi-infinite cone, calculated from the exact solution of Stokes' equations and according to the similarity assumption ('tube flow') for different values of the cone half-angle α.

deviations between the exact solution and the similarity assumption ('Poiseuille flow') are visible.

A similar procedure has been carried through for the cone flow. The similarity assumption predicts the following stream function:

$$\Psi_{\mathrm{SA}}(\eta) = \frac{Q}{2\pi} \cdot \eta^2 \cdot \left(2 - \eta^2\right) \tag{49}$$

with

$$\eta = \frac{r}{R(z)} = \frac{\tan\vartheta}{\tan\alpha}. \tag{50}$$

The result is shown in Fig. 7 as a plot of the dimensionless stream function $F = \pi\Psi/Q$ calculated from Eq. (49) and the exact solution calculated following Eq. (43) versus η for different values of the cone half-angle α. Again, up to $\alpha = 15°$ there is excellent agreement between the two stream functions.

6. Conclusions

Forced convective heat transfer problems in ducts with slowly varying cross-section can be reduced to the analogous problems for uniform duct cross-section, if the flow kinematics obeys the similarity assumption, i.e. the axial velocity profile is locally fully developed. The latter assumption holds, of course, never exactly but only within certain limits of accuracy. These limits have been discussed for creeping flow of a Newtonian liquid in simplified geometries. For non-Newtonian flow and more complicated geometries (non-uniform wall inclination angle) there is no direct means of proving the accuracy of the assumed kinematics, apart from numerical simulation.

The laminar forced convective heat transfer in a circular duct with weakly non-uniform radius is invariant with respect to the tube radius. It is only dependent on the tube Fourier number $Fo = \pi aL/Q$. Solutions to generalised Graetz problems derived for the case of a uniform tube radius may also be applied if the tube radius is weakly non-uniform. This result holds exactly for any (axisymmetric) shape of the velocity profile, provided the similarity assumption is justified.

For flow in a plane channel with slowly varying distance between the channel walls the fluid bulk temperature can be shown to depend only upon the effective Fourier number $Fo_{\mathrm{eff}} := (aB/Q) \cdot \int_0^L (\mathrm{d}z/H(z))$. The functional dependence of the bulk temperature on Fo_{eff} is the same as for a channel with uniform height. Hence, solutions derived for uniform height may also be applied when the distance between the plates is non-uniform. The effective Fourier number comprises an integral over the reciprocal value of the local channel height. It is thus seen, that small values of the local channel height are advantageous for the heat transfer process. This result has been well known for the case of a uniform channel. It has now been generalised for a weakly non-uniform channel.

It is emphasised that, using the analogy between heat and mass transfer, all results derived in this paper may as well be applied to forced convective mass transfer in ducts with non-uniform cross-sections.

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